Opportunity #9: Calculus

Mar. 8, 2018

ANSWERS

Routine Questions

1. Find each derivative, $\frac{dy}{dx}$:
   a. $y = \sin x + 3x^4 - x + 1$

      $\frac{dy}{dx} = \cos x + 12x^3 - 1$

   b. $y = \sqrt[3]{\frac{1}{x^4}}$

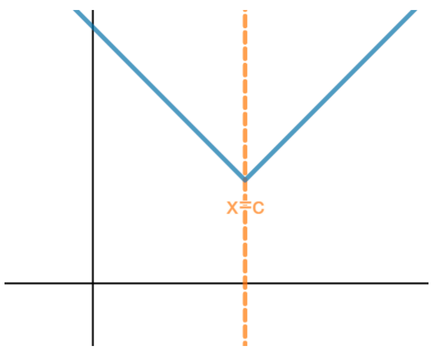
      $y = x^{-\frac{4}{3}}$

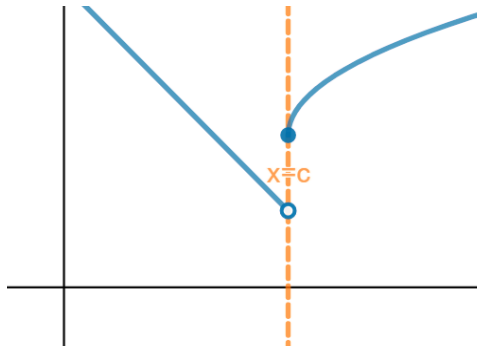
      $\frac{dy}{dx} = -\frac{4}{3}x^{-\frac{7}{3}} = -\frac{4}{3\sqrt[3]{x^7}}$

2. Use the definition of derivative as a limit to find $f'(x)$ when $f(x) = 3x^2 - 7$.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 7] - [3x^2 - 7]}{h} \\
&= \lim_{h \to 0} \frac{[3(x^2 + 2xh + h^2) - 7] - [3x^2 - 7]}{h} \\
&= \lim_{h \to 0} \frac{[3x^2 + 6xh + 3h^2 - 7] - [3x^2 - 7]}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 7 - 3x^2 + 7}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h \to 0} (6x + 3h) \\
&= 6x + 3(0) = 6x
\end{aligned}
$$

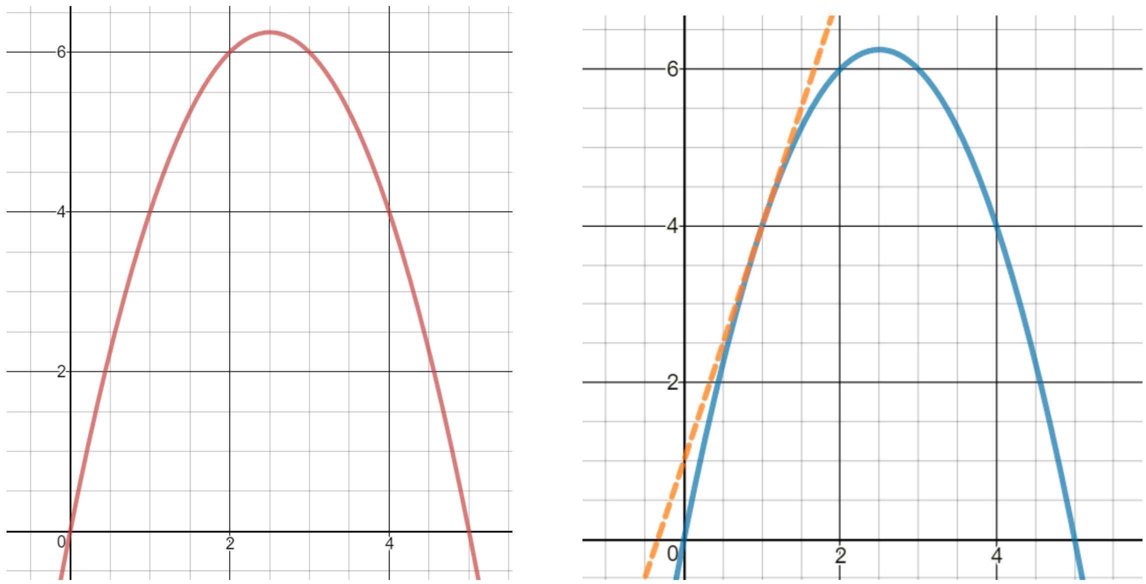
3. Sketch two *different* examples where $y = f(x)$ is defined at $x = c$ but the function is not differentiable at $x = c$.
Answers may vary.





4. What can we say about $f'(x)$ and/or $f''(x)$ when the graph of $f(x)$...
    a. is increasing?
    $f'(x) > 0$
    b. is decreasing?
    $f'(x) < 0$
    c. is concave up?
    $f''(x) > 0$, $f'(x)$ ***is increasing***
    d. is concave down?
    $f''(x) < 0$, $f'(x)$ ***is decreasing***
    e. reaches the top (local max) of a smooth, continuous curve?
    $f'(x) = 0$ ***and is going from positive to negative***
    $f''(x) < 0$
    f. reaches the bottom (local min) of a smooth, continuous curve?
    $f'(x) = 0$ ***and is going from negative to positive***
    $f''(x) > 0$
    g. reaches an inflection point?
    $f''(x) = 0$ ***and is going from positive and negative or vice versa***
    $f'(x)$ ***is changing between increasing and decreasing or vice versa***

5. Suppose the following graph represents the height of an object in meters above the ground as measured over time in seconds. What is the velocity at $t=1$ second?



The slope of the tangent line is 3. Therefore the velocity is 3 m/s

Thinking Questions

6. Explain what differentiability is, how a function might fail to be differentiable, and why it matters. (For maximum points, your writing should be clear, precise, and thorough; don't be vague.)
Differentiability means the derivative of a function exists. In order to be differentiable at a certain point, the function must be continuous at that point and the limit as x approaches that point of f' must exist. For the limit of f' to exist, the slope of the tangent line must approach the same value as you approach the point from the left and the right. Some of the cases where the function fails to be differentiable are when the function features a sharp corner, called a cusp, when it has a vertical tangent line (because the slope of a vertical line is undefined), and when the function has a discontinuity. Differentiability is important because it allows us to use derivatives, which are a powerful tool for analysing anything related to the rate of change of the function.